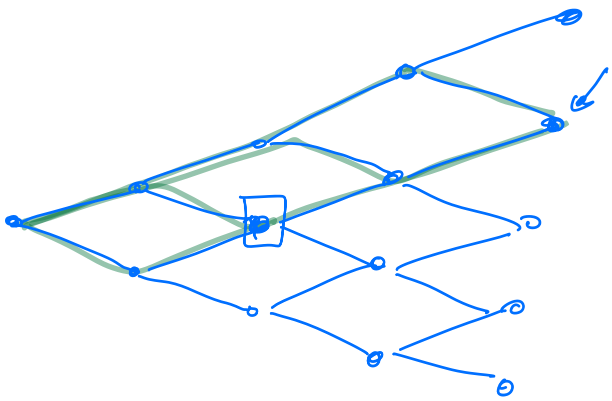
Summary: Normal distribution

Brownian Motion; Exercises

Motivation: going back to the binomial model, let's have a look at the binomial tree:



Binomial because at time n, the number of ups in the stock price is Binomial$\left(n, \frac{1+R-d}{u-d}\right)$

Imagine that you consider time steps with smaller and smaller amplitude

$\text{Bin}(n, q_u) \xrightarrow[n\to\infty]{} \mathcal{N}(\mu, \sigma^2)$

(binomial distribution) (normal distribution)

$\Downarrow$ $\Downarrow$

Binomial Model Brownian Motion

Before defining what is a Brownian Motion, we will recall some basics related with normal distribution.

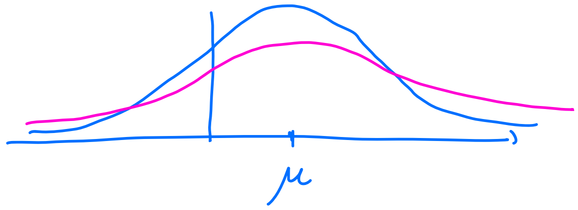
Normal distribution

X (a random variable) is normally distributed with expected value μ and variance σ^2 if:

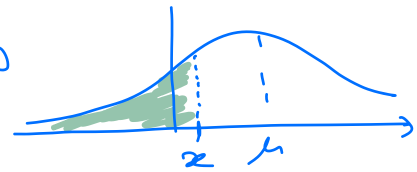
- $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $x \in \mathbb{R}$

$X \sim \mathcal{N}(\mu, 1)$

$Y \sim \mathcal{N}(\mu, 2)$



- $P(X \le x) = \int_{-\infty}^{x} f_X(s)\, ds$



- $E[X] = \mu$ ("mean": expected value)

- $var[X] = \sigma^2$

One particular case: standard normal distribution

$$\mathcal{N}(0,1)$$

Result: $X \sim \mathcal{N}(\mu, \sigma^2)$

$$\iff$$

$$Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$$

— " —

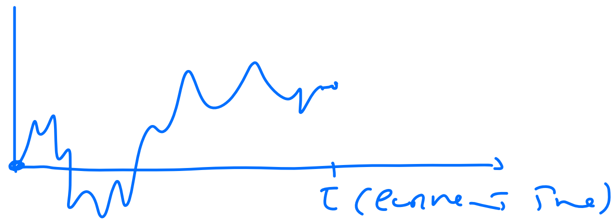
Returning to the Brownian motion:

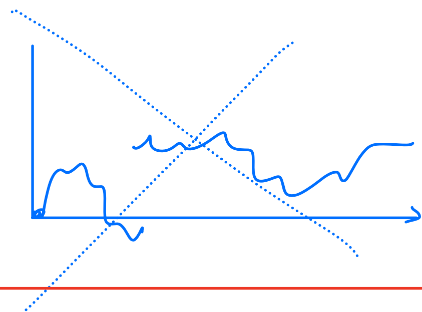
<u>Definition</u>: $\{W(t),\ t \ge 0\}$ is a Brownian motion if the following properties hold:

i) $W(0) = 0$

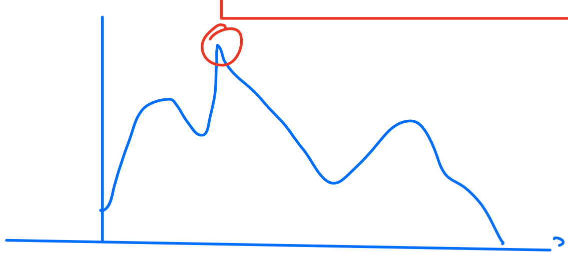
ii) All the sample paths are continuous functions.

sample-path: is a possible realization of the process (it is what I observed from a process). It is no longer a random variable; it is the information that I have from time 0 to current time.



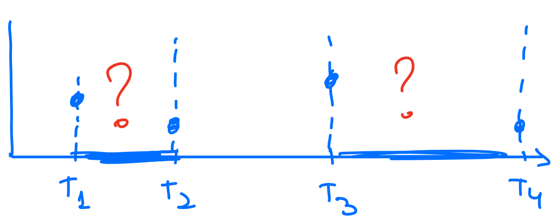


iii) Although all the sample-paths are continuous, they are non-differentiable everywhere.



iv) $W(t) \sim \mathcal{N}(0, \overset{\sigma^2}{t})$ $\to$ $E[W(t)] = 0$, $Var(W(t)) = t, \ \forall t$

v) independent increments

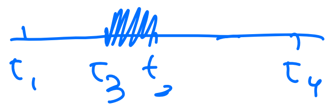


non-overlapping intervals

$$(W(t_2) - W(t_1)) \perp\!\!\!\perp (W(t_4) - W(t_3))$$

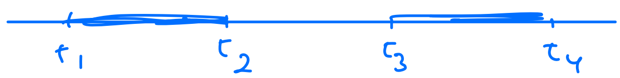
$$f_{W(t_2)-W(t_1),\,W(t_4)-W(t_3)}(a,b) =$$

$$f_{W(t_2)-W(t_1)}(a)\; f_{W(t_4)-W(t_3)}(b)$$



vi) stationary increments

$t_2 - t_1 = \tau \qquad t_4 - t_3 = \tau$



$W(t_2) - W(t_1) \sim \mathcal{N}(0, \tau)$

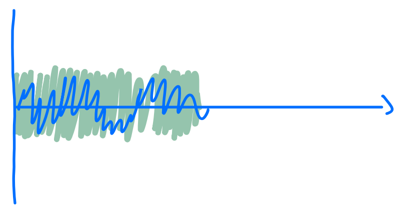
$W(t_4) - W(t_3) \sim \mathcal{N}(0, \tau)$

$W(t) - W(s) \sim \mathcal{N}(0, t-s), \ \forall s \le t$

$$\big(W(t_2) - W(t_1)\big) \overset{D}{=} \big(W(t_4) - W(t_3)\big)$$

Some consequences of the definition / Remember...

- $E[W(s)] = 0, \ \forall s$

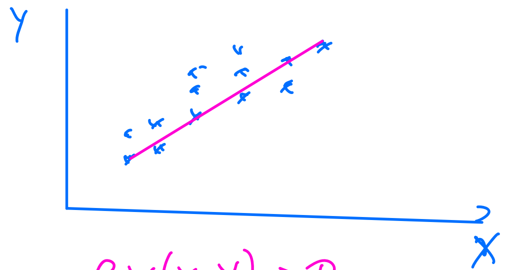


- $\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$

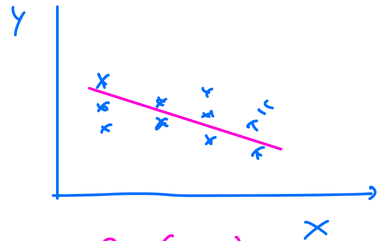
  if X and Y are independent random variables, $\mathrm{Cov}(X, Y) = 0$

  but

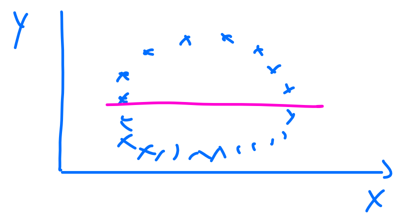
  $\mathrm{Cov}(X, Y) = 0$ does not imply independence!



$Cov(X,Y) > 0$



$Cov(X,Y) < 0$



$Cov(X,Y) = 0$ but they are not independent

So in general independence is not equivalent to zero covariance EXCEPT if X and Y are NORMALLY DISTRIBUTED

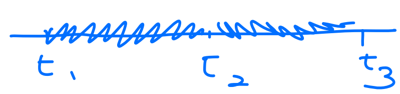
$$Cov(X,Y) = 0 \Leftrightarrow X \perp\!\!\!\perp Y \quad \text{(normal case)}$$

Coming back to the Brownian motion, as it has independent increments, it means that:

$$Cov\big(W(t_2) - W(t_1), W(t_3) - W(t_2)\big) = 0$$

$$\Leftrightarrow W(t_2) - W(t_1) \perp\!\!\!\perp W(t_3) - W(t_2)$$

↳ "independent"



3. Regarding the Brownian motion:

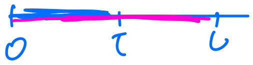
   a) Prove that for $0 < t < u$, $\text{Cov}(W(t) - \frac{t}{u}W(u), W(u)) = 0$.

   b) Is $W(t) - \frac{t}{u}W(u)$ independent of $W(u)$? Justify your answer.

   c) Prove that

$$\mathbb{E}[W(t)|W(u)] = \frac{t}{u}W(u).$$

a) $0 < \tau < u$: $\quad \operatorname{cov}\left(W(\tau) - \frac{\tau}{u} W(u), W(u)\right) = 0$



$\operatorname{cov}(W(\tau) - \underset{0}{\underset{\shortparallel}{W(0)}}, W(u) - W(\tau)) = 0$

(due to independent increments)

$$\operatorname{cov}\left(W(\tau) - \frac{\tau}{u} W(u), W(u)\right) = \underbrace{\operatorname{cov}(W(\tau), W(u))}_{a} + \underbrace{\operatorname{cov}\left(-\frac{\tau}{u} W(u), W(u)\right)}_{b}$$

$[\operatorname{cov}(X+Y, Z) = \operatorname{cov}(X, Z) + \operatorname{cov}(Y, Z)]$

a) $\operatorname{cov}(W(\tau), W(u)) = \operatorname{cov}(W(\tau), W(u) - W(\tau) + W(\tau))$

$= \operatorname{cov}(W(\tau), W(u) - W(\tau)) + \operatorname{cov}(W(\tau), W(\tau))$

$W(\tau) \perp\!\!\!\perp (W(u) - W(\tau))$

$= 0 + \operatorname{var}(\underbrace{W(\tau)}_{N(0,\tau)}) = \tau \qquad [\operatorname{cov}(X, X) = \operatorname{var}(X)]$

In general: $\operatorname{cov}(W(\tau), W(u)) = \tau$,
$(\tau \leq u)$

b) $\operatorname{cov}\left(-\frac{\tau}{u} W(u), W(u)\right) = -\frac{\tau}{u} \operatorname{cov}(W(u), W(u))$

$= -\frac{\tau}{u} \operatorname{var}(W(u)) = -\frac{\tau}{u} \times u = -\tau$

$[\operatorname{cov}(aX, Y) = a \operatorname{cov}(X, Y)]$

Therefore

$\operatorname{cov}\left(W(\tau) - \frac{\tau}{u} W(u), W(u)\right) = \tau - \tau = 0$

b) Is $W(\tau) - \frac{\tau}{u} W(u)$ independent of $W(u)$?

Yes, because we have two normal random variables and in that case zero covariance is the same as independence.

e) $\mathbb{E}[W(t)|W(u)] = \frac{t}{u}W(u).$

$= E\left[W(t) - \frac{t}{u}W(u) + \frac{t}{u}W(u) \,\middle|\, W(u) \right]$

$W(t) - \frac{t}{u}W(u)$ independent of $W(u)$?

$= \underbrace{E\left[W(t) - \frac{t}{u}W(u) \,\middle|\, W(u) \right]}_{\text{independence}} + E\left[\frac{t}{u}W(u) \,\middle|\, W(u) \right]$

$= E\left[W(t) - \frac{t}{u}W(u) \right] + \frac{t}{u} E\left[W(u) \,|\, W(u) \right]$

$= E[W(t)] - \frac{t}{u}E[W(u)] + \frac{t}{u}W(u)$

$= 0 - \frac{t}{u} \times 0 + \frac{t}{u}W(u) = \frac{t}{u}W(u)$ //

5. Let $W = \{W(t), t \geq 0\}$ be a Brownian motion and $S = \{S(t), t \geq 0\}$, where $S(t) = e^{\mu t + \sigma W(t)}$, with $S(0) = 1$.

   a) Derive an expression for $P(S(t) \leq x)$.

   b) Determine an expression for the conditional expectation $\mathbb{E}[S(t)|\mathcal{F}_s]$, where $s < t$ and $\{\mathcal{F}_s, s \geq 0\}$ is the filtration associated with the process S.

   c) Find conditions on μ and σ under which the process $\{S(t), t \geq 0\}$ is a martingale.
   Note: You may need to use the following fact: the moment generating function of a random variable Y with normal distribution, with parameters μ and σ is given by:

$$\mathbb{E}[e^{sY}] = e^{s\mu + \frac{1}{2}\sigma^2 s^2}, \quad s \in \mathbb{R}.$$

a) $P(S(t) \leq x) = P\left(e^{\mu t + \sigma W(t)} \leq x\right)$

$= P\left(\mu t + \sigma W(t) \leq \ln x\right) =$

$= P\left(W(t) \leq \frac{\ln x - \mu t}{\sigma}\right) = \underbrace{F_{\mathcal{N}(0,t)}\left(\frac{\ln x - \mu t}{\sigma}\right)}$

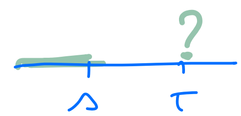
$W(t) \sim \mathcal{N}(0,t)$

distribution function of a normal random variable with expected value 0 and variance t at the point $\frac{\ln x - \mu t}{\sigma}$

b) $E[S(t) \,|\, \mathcal{F}^W_s] = E\left[e^{\mu t + \sigma W(t)} \,\middle|\, \mathcal{F}^W_s\right]$

$s < t$

$$= e^{\mu t} E[e^{\sigma W(t)} \mid \mathcal{F}_s^W]$$



$$= e^{\mu t} E[e^{\sigma(W(t)-W(s)) + \sigma W(s)} \mid \mathcal{F}_s^W]$$

$$= e^{\mu t} E[e^{\sigma(W(t)-W(s))} \cdot \underbrace{e^{\sigma W(s)}} \mid \mathcal{F}_s^W]$$

$$= e^{\mu t} e^{\sigma W(s)} E[e^{\sigma(W(t)-W(s))} \mid \mathcal{F}_s^W]$$

$$= e^{\mu t} e^{\sigma W(s)} E[e^{\sigma(W(t)-W(s))}]$$

$$= e^{\mu t + \sigma W(s)} E[e^{\sigma Y}] \qquad X \sim N(0, t-s)$$

$$\mathbb{E}[e^{sY}] = e^{s\mu + \frac{1}{2}\sigma^2 s^2}, \quad s \in \mathbb{R}.$$

$Y \sim N(\mu, \sigma^2)$

$$\begin{cases} s = \sigma \\ \mu = 0 \\ \sigma^2 = t - s \end{cases}$$

$$= e^{\mu t + \sigma W(s)} e^{\sigma \times 0 + \frac{1}{2}(t-s)\sigma^2} = e^{\mu t + \frac{1}{2}(t-s)\sigma^2 + \sigma W(s)}$$

c) values for μ and σ such that $\{S(t), t \geq 0\}$ is a martingale

$$\underbrace{E[S(t) \mid \mathcal{F}_s^W]}_{e^{\mu t + \frac{1}{2}(t-s)\sigma^2 + \sigma W(s)}} = \underbrace{S(s)}_{e^{\mu s + \sigma W(s)}}, \quad \forall s \leq t$$

So it will be a martingale if and only if:

$$\mu t + \frac{1}{2}(t-s)\sigma^2 = \mu s, \quad \forall t, s$$

18. For $a > 0$, define a the process $\tilde{W} = \{\tilde{W}(t), t \geq 0\}$, where $\tilde{W}(t) = \frac{1}{\sqrt{a}} W(at)$. Show that $\tilde{W}$ is also a Brownian motion.

- $\tilde{w}(0) = 0$ $\qquad \tilde{w}(0) = \frac{1}{\sqrt{a}} w(0) = 0$ ✓

- Continuity of the sample paths and non-differentiability : we skip this.

- $\tilde{w}(t) \sim \mathcal{N}(0, t)$? ✓

$\tilde{w}(t) = \frac{1}{\sqrt{a}} w(at)$ $\qquad w(at) \sim \mathcal{N}(0, at)$

$\frac{1}{\sqrt{a}} w(at) \sim \mathcal{N}\left(0, \frac{at}{a}\right) =$

$= \mathcal{N}(0, t)$

$\left[\begin{array}{l} X \sim \mathcal{N}(\mu, \sigma^2) \\ aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2) \end{array}\right]$

- $\operatorname{cov}\left(\tilde{w}(s), \tilde{w}(t) - \tilde{w}(s)\right) = \operatorname{cov}\left(\frac{1}{\sqrt{a}} w(as), \frac{1}{\sqrt{a}} w(at) - \frac{1}{\sqrt{a}} w(as)\right) =$

$= \frac{1}{\sqrt{a}} \operatorname{cov}\left(w(as), w(at) - w(as)\right) = 0$

Thus we have independence of the increments

- $\tilde{w}(t) - \tilde{w}(s) \sim \mathcal{N}(0, t-s)$ ($\Leftrightarrow$ stationarity)

$\tilde{w}(t) - \tilde{w}(s) = \frac{1}{\sqrt{a}} w(at) - \frac{1}{\sqrt{a}} w(as)$

$$= \frac{1}{\sqrt{a}} \Big(\underbrace{W(\overset{\tau}{\widetilde{at}}) - W(\overset{s}{\widetilde{as}})}_{N(0,\ at-as)} \Big) = \frac{1}{\sqrt{a}} X \sim N(0, t-s)$$

with $X \sim N(0, at-as) = N(0, a(t-s))$

[$W(\tau) \sim N(0,\tau)$; $W(\tau) - W(s) \sim N(0, t-s)$]

Therefore $\frac{1}{\sqrt{a}} X \sim N\Big(0, \frac{a(t-s)}{(\sqrt{a})^2}\Big) = N(0, t-s)$

[$X \sim N(\mu, \sigma^2)$ Then $aX+b \sim N(a\mu+b, a^2\sigma^2)$]